Tutorial 3

Sum of combinatorial games

Let $G_1, \cdots, G_n$ be n combinatorial games. Let G denote their sum. Let $g_1, \cdots, g_n$ be the S-G functions of $G_1, \cdots, G_n$ respectively and let g be the S-G function of G.

Proposition 1.

$$g(x_1, \cdots, x_n) = g_1(x_1) \oplus \cdots \oplus g_n(x_n).$$

Exercise 1. *Consider the following* 3 *games.*

G_1*:* 1*-pile nim.*

G_2*: Subtraction game with* $S = \{1, 2, 3, 4, 5, 6\}$.

G_3*: When there are* n *chips remaining, a player can remove only* 1 *chip if* n *is odd and can remove any positive even number of chips if* n *is even.*

Let g_1, g_2, g_3 *be the S-G functions of the* 3 *games respectively. Let* G *denote the the sum of* G_1, G_2, G_3 *and let* g *be the S-G function of* G.

(i) Find $g_1(14), g_2(20), g_3(24)$.

(ii) Find $g(14, 20, 24)$.

(iii) Find all winning moves of G *with position* $(14, 20, 24)$.

Solution: (i) Since G_1 is 1-pile nim, we have $g_1(n) = n$ for all n, hence $g_1(14) = 14$. Since G_2 is subtraction game with $S = \{1, 2, 3, 4, 5, 6\}$, we have $g_2(20) = 6$ since $20 \equiv 6(\text{mod}7)$. To find g_3, by backwards induction, we have

k	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$g_3(k)$	0	1	1	0	2	0	3	0	4	0	5	0	6	$\cdots$

Hence we have

$$g_3(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 3 \text{ and } k \text{ is } \textit{odd} \\ \frac{k}{2} & \text{if } k \text{ is even} \end{cases}$$

Hence $g_3(24) = 12$.

(ii) By (i), we have

$$g(14, 20, 24) = g_1(14) \oplus g_2(20) \oplus g_3(24) = 14 \oplus 6 \oplus 12 = 4.$$

(iii) Since

$$14 \oplus 6 \oplus 12 = \begin{array}{ll} (1,1,1,0)_2 & \\ (0,1,1,0)_2 & \\ (1,1,0,0)_2 & \\ \hline (0,1,0,0)_2 & =4 \end{array}.$$

All winning moves are: choosing G_1 and removing 4, or choosing G_2 and subtracting 4, or choosing G_3 and removing 4 chips.

Two-person zero-sum games

Definition 1. *A game is called a* two-person zero-sum game *if*

(i) Two players make their moves simultaneously.

(ii) One player wins what the the other player loses.

Strategic form

Definition 2. *A* strategic form *of a two-person zero-sum game is a triple (X, Y, π), where X, Y are the sets of strategies of Player I and Player II respectively, and $\pi : X \times Y \to \mathbb{R}$ is the payoff function of Player I.*

In this note, we only consider the case that both X and Y are finite, so that we can identify the payoff function as a matrix.

Matrix game

Assume $X = \{1, \cdots, m\}$, $Y = \{1, \cdots, n\}$ *are the sets of strategies of Player I (the row player) and Player II (the column player) respectively. Let* $A \in M_{m\times n}(\mathbb{R})$ *be the payoff matrix, that is,* $a_{i,j}$ *denotes the payoff of the the row player when the row player takes his strategy* i *and the column player takes his strategy* j.

Pure strategy: If A has a saddle point $a_{k,l}$, that is

$$a_{k,l} = \min_{1\leq j\leq n} a_{k,j} = \max_{1\leq i\leq m} a_{i,l},$$

then the row player has an optimal pure strategy k and the column has an optimal pure strategy l.

Mixed strategy: Let $\mathcal{P}^m$ denote the collection of p dimensional probability vectors. We call each probability vector $\boldsymbol{p} \in \mathcal{P}^m$ a mixed strategy for the row player. Similarly, each $\boldsymbol{q} \in \mathcal{P}^n$ is called a mixed strategy for the column player.

Theorem 2. *(Minimax Theorem). Let* A *be an* $m \times n$ *matrix. Then there exist a number* $v \in \mathbb{R}$ *and two probability vectors* $\boldsymbol{p} \in \mathcal{P}^m$, $\boldsymbol{q} \in \mathcal{P}^n$ *such that*

(i) $\boldsymbol{p}A\boldsymbol{y}^T \geq v$ *for any* $\boldsymbol{y} \in \mathcal{P}^n$.

(ii) $\boldsymbol{x}A\boldsymbol{q}^T \leq v$ *for any* $\boldsymbol{x} \in \mathcal{P}^m$.

(iii) $\boldsymbol{p}A\boldsymbol{q}^T = v$.

Remark: (1) The number v in the above theorem is unique, and we call it the value of A, write $v = v(A)$.

(2) In the above theorem, we call $\boldsymbol{p}$ an optimal (mixed) strategy for the row player and $\boldsymbol{q}$ an optimal (mixed) strategy for the column player. In general, $\boldsymbol{p}$ and $\boldsymbol{q}$ may not be unique.

(3) If $v = 0$, we say this game is fair.

(4) By solving a matrix game, we mean finding the value of matrix A and optimal strategies for the two players.

Exercise 2. *Show that the number v in the Minimax Theorem is unique.*

Proof. Suppose two triples $(v, \boldsymbol{p}, \boldsymbol{q})$, $(v', \boldsymbol{p}', \boldsymbol{q}')$ both satisfy (i), (ii), (iii) in the Minimax Theorem. Note that by using (i), (ii) several times, we have

$$v \leq \boldsymbol{p}A\boldsymbol{q}'^T \leq v' \leq \boldsymbol{p}'A\boldsymbol{q}^T \leq v.$$

Exercise 3. *Prove if $A^T = -A$, then $v(A) = 0$.*

Proof. Write $v(A) = v$. Assume $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}^n$ are optimal strategies. Then by the Minimax Theorem, we have

$$\begin{cases} \boldsymbol{p}A\boldsymbol{y}^T \geq v, & \forall \boldsymbol{y} \in \mathcal{P}^n. \\ \boldsymbol{x}A\boldsymbol{q}^T \leq v, & \forall \boldsymbol{x} \in \mathcal{P}^n. \\ \boldsymbol{p}A\boldsymbol{q}^T = v. \end{cases}$$

Taking transpose in the above equations and applying the assumption that $A^T = -A$, we have

$$\begin{cases} \boldsymbol{y}A\boldsymbol{p}^T \leq -v, & \forall \boldsymbol{y} \in \mathcal{P}^n. \\ \boldsymbol{q}A\boldsymbol{x}^T \geq -v, & \forall \boldsymbol{x} \in \mathcal{P}^n. \\ \boldsymbol{q}A\boldsymbol{p}^T = -v. \end{cases}$$

By the Minimax Theorem and the uniqueness of the value of A, we have $v = -v$, hence $v = 0$.

Solving matrix games

Two useful principles: 1. Deleting the dominated rows and columns to obtain a new matrix with lower dimensions. Recall that a row is dominated if it is dominated (or say bounded) from above by another row, a column is dominated if it is dominated from below by another column.

2. The principle of indifference. Assume $\boldsymbol{p} = (p_1, \cdots, p_m)$ and $\boldsymbol{q} = (q_1, \cdots, q_n)$ are optimal strategies for Player I and Player II respectively. Then

(i) for any $k \in \{1, \cdots, m\}$ with $p_k > 0$, we have $\sum_{j=1}^{n} a_{k,j} q_j = v(A)$.

(ii) for any $l \in \{1, \cdots, n\}$ with $q_l > 0$, we have $\sum_{i=1}^{m} a_{i,l} p_i = v(A)$.

Exercise 4. *In a Rock-Paper-Scissors game, the loser pays the winner an amount of money which is equal to the total number of fingers shown by the two players (for example, if Player I shows Scissors and Player II shows Paper, then Player II should pay* 7 *dollars to Player I).*

(i) Find the value of the games.

(ii) Find optimal strategies for the two players.

Exercise 5. *Let*

$$A = \begin{pmatrix} 0 & -2 & 2 & 1 & 4 \\ 2 & -1 & 3 & 0 & 5 \\ 3 & 4 & -2 & 5 & -3 \end{pmatrix}$$

(i) Find the reduced matrix of A by deleting dominated rows and columns.

(ii) Solve the two-person zero-sum game with game matrix A.